Development of Axisymmetric Green's Function for an Exponentially Graded Inhomogeneous Elastic Solid

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Abstract

The present work attempts to determine the Green's function for an inhomogeneous elastic solid (also known as functionally graded material) subjected to body forces in cylindrical coordinates. Solutions are developed using classical techniques such as, separation of variables and integral transforms. The difficulties associated with integral transforms are highlighted and certain approximations are attempted to obtain solutions. The Green's function forms the most important part in developing solutions to boundary value problems using the boundary element method (BEM).

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Contents

Chapter 1

Introduction

1.1 Functionally Graded Materials

In the mid 1980's, Japanese researchers working on a hypersonic space plane project developed a new class of materials called functionally graded materials (FGM) [1]. The defining characteristic of FGMs is the continuous variation in material properties over a certain dimension of the solid instead of having, either the same properties throughout, or being bonded to another material.

Typical properties that might vary in FGMs can range from mechanical properties such as Young's modulus, shear modulus,thermal conductivity, or magnetic properties [2]. The variation of these properties is usually modeled as either linear or exponential with respect to position, as it makes for easier mathematical analysis. While originally FGMs were designed for use in the aerospace industry, they were later adopted for use in the biomedical, automotive, defense, and energy industries. As FGMs have become more popular, more demand has grown for analysis of the behavior of FGMs under different kinds of loadings and for different shapes. This analysis is especially simplified if the Green's function for an FGM under a certain load can be determined

1.2 Green's Functions

In the theory of ordinary and partial differential equations, the Green's function is the solution to a differential equation with a unit impulse input. While typically the differential equation corresponding to the Green's function is linear, the Green's function for a non-linear differential equation can also be obtained [3]. The main appeal of the Green's function is that the product of the Green's function and an arbitrary input can be integrated to find the solution with an arbitrary input. A standard example is the deflection

of a simply supported beam under a generic transverse loading. The governing equation given by Euler Bernoulli beam theory is $EIv'''' = -w(x)$. If $w(x)$ is replaced with an impulse force which is modeled as a Dirac delta function, $\delta(x)$, then the solution of that equation becomes the Green's function. The results from integrating the Green's function for a beam under different loads are commonly displayed in tables in textbooks and taught in undergraduate mechanics of materials courses.

In boundary value problems of elasticity, the Green's function, also known as Kelvin's solution. Kelvin's solution is necessary in the development of boundary element method (BEM) solutions. BEM is a method of numerically solving partial differential equations similar to that of finite element method (FEM). In BEM solutions,however, the partial differential equation is formulated as an integral equation and only requires the discretization of the boundary instead of the entire volume. Due to this, BEM solutions typically are more efficient computationally, with a reduction in the dimensionality, and can effectively handle boundary conditions at infinity in comparison to FEM [4]. Since the development of FGMs, Green's functions for various geometries and styles of grading have been sought in order to aid in the development of BEM solutions.

1.3 Review of Relevant Literature

Since the demand for Green's functions for FGMs has gone up, several have been found and published. Recently, Watanabe [5] solved for the Green's function for a two dimensional unidirectional exponentially graded solid in Cartesian coordinates by means of a double Fourier transform. Upon the Fourier transformation, the original two partial differential equations were transformed into two algebraic equations which were then able to be solved and inverted again yielding the Green's function. In a similar fashion, Martin et. al [6] obtained the Green's function for a three dimensional solid with exponential grading in three directions. Hasegawa [7] solved the Green's function for a homogeneous cylinder under a torsional load by using what are known as stress functions. Mokashi [8] solved for the torsional Green's function for an infinite cylinder with exponential grading in both the radial and longitudinal direction. The method of solution was to use Hankel transforms in the case of longitudinal grading and Fourier transforms in the case of radial grading. While the solution for the longitudinal grading was presented in exact closed form as an in integral, the integral for inverting the transformed solution of the equations is not standard. Instead, pertubation methods were used to solve for the Green's function in the case of small values of radial inhomogeneity.

Filon [9] obtained the components of the displacement vector field of a homogeneous (non-graded) cylinder subjected to normal radial pressures and tangential shears. The form of the equations closely resembled the governing equations for the Green's function for a exponentially graded cylinder. Filon's method of solution

was to recognize that the derivatives of the components of the displacement vector field satisfied a partial differential equation which had a well known solution consisting of trigonometric and Bessel functions. With this, the components of the displacement vector field could be obtained by integrating Bessel functions and solving for the appropriate constants.

1.4 Purpose

This thesis aims to provide the reader with the generalized procedure to obtain a Green's function in cylindrical coordinates to the problem of an infinite non-homogeneous elastic space subject to ring type impulse body forces. The development of this methodology starts with the basics of tensors and their use in the formulation of the governing equations. Solution techniques using integral transforms are then discussed for certain example problems. The limitations associated with these techniques will also be discussed in detail. Certain assumptions can be made which remove these limitations and allow for standard techniques such as integral transforms. The physical interpretations of these assumptions are discussed.

Chapter 2

Basics of Tensors

Tensors are fundamental to the understanding of the governing equations of elasticity. A brief exposition on tensors and tensor calculus will be given in this section. The machinery developed in this section will then be used to derive the governing equations of elasticity. The definition given for a tensor from Wolfram Mathworld [10] is "an n-th rank tensor in m dimensional space is an object that that has n indices and m^n components and obeys certain transformation laws." For example, a rank zero tensor is a scalar ie a real number which has no indices. A rank one tensor is a vector with one index and the number of components is equal to the dimension being considered which is generally equal to three. A rank two tensor, such as the Cauchy stress tensor,the strain tensor, or the moment of inertia tensor has two indices and three dimensions has 9 components. Tensors such as those listed can be thought of as a generalized representation of a physical quantity, The components of any tensor are written using subscripts and superscripts. The components of a rank two tensor T may be written as $T_{ij},\ T^{ij},$ or T_i^j. The choice of subscripts or superscripts is dependent on which type of transformation law the tensor obeys. The two types of transformation laws for vectors and tensors are known as *covariant* and *contravariant* transformation laws.

2.1 Covariant and contravariant vectors and tensors

Consider two different coordinate systems (x^1, x^2, x^3) and $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$. where

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3)$$

and

$$x^i = x^i(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$$

If the components of a vector in the x_i coordinate system are described by T_j then the vector is said to be *covariant* if the components can be transformed to the $\tilde{x}_i$ coordinate system by means of the transformation

$$\tilde{T}_i = \sum_{j=1}^{3} \frac{\partial x^j}{\partial \tilde{x}^i} T_j \tag{2.1}$$

By convention, covariant vectors and tensors use subscripts to denote their components. *Contravariant* vectors and tensors, whose components are denoted using superscripts, obey the transformation law

$$\tilde{T}^i = \sum_{j=1}^{3} \frac{\partial \tilde{x}^i}{\partial x^j} T^j \tag{2.2}$$

A tensor of rank two or higher is said to be *mixed* if it has both covariant and contravariant components. A mixed tensor T_i^j obeys the transformation law

$$\tilde{T}_m^n = \sum_{m=1}^{3} \sum_{k=1}^{3} \frac{\partial x^i}{\partial \tilde{x}^m} \frac{\partial \tilde{x}^n}{\partial x^j} T_i^j \tag{2.3}$$

The components of the Cauchy stress tensor and the strain tensor which are used to derive the governing equations of elasticity are both second order covariant tensors. The components of the stress tensor are written σ_{ij} and the components of the strain tensor are written ε_{ij}.

2.2 The Einstein summation convention

Part of the convenience of using tensors is the ability to express complex equations in a relatively simple form. In order to do this, a notation convention known as the *Einstein summation convention* was introduced. The convention is used to imply a summation over an index by repeating indexes. For example, the dot product of two vectors, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is usually written as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i \tag{2.4}$$

Using the summation convention, the summation symbol in equation 2.1 is implied and can be omitted which simplifies the expression to

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i$$

Using this convention, equations 2.1-2.3 can be written more concisely. For example equation 2.1 becomes

$$\tilde{T}_i = \frac{\partial x_j}{\partial \tilde{x}_i} T_j \tag{2.5}$$

The ∇ operator in Cartesian coordinates is also able to be written concisely using this notation

$$\nabla = \frac{\partial}{\partial x^1}\hat{e}_1 + \frac{\partial}{\partial x^2}\hat{e}^2 + \frac{\partial}{\partial x^3}\hat{e}_3 = \frac{\partial}{\partial x^i}\hat{e}_i$$

2.3 The Kronecker delta

An important function that is commonly used in tensor calculus is known as the *Kronecker delta function*. The Kronecker delta function has two indices and is equal to one if the indices are equal and equal to zero if the indices are not equal. More formally

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In matrix form, the Kronecker delta is given by the matrix

$$I_3 = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The utility of the Kronecker delta is that it allows for the changing of indices of tensors. Let v_i be a vector or rank one tensor. If we multiply v_i by δ_{ij} we have $v_i\delta_{ij}$. As the i index is repeated, this implies a sum over i and that j is a free index. Say, without loss of generality, that $j = 2$. then, performing the summation,

$$v_i\delta_{ij} = v_1\delta_{12} + v_2\delta_{22} + v_3\delta_{32}$$

However, $\delta_{12} = \delta_{32} = 0$ while $\delta_{22} = 1$. This implies $v_i\delta_{ij} = v_2\delta_{22} = v_2 = v_j$. Hence, the i index in v_i has been replaced with a j . This result can be applied to tensors of higher order. For example, for a second order covariant tensor T_{ij}

$$T_{ij}\delta_{ik} = T_{kj}$$

The utility of the Kronecker delta will eventually be seen in the derivations of the governing equations of

elasticity.

2.4 The Permutation symbol

Much like the Kronecker delta, another tensor that comes up frequently in tensor analysis is known as the *Permutation* also known as the Levi-Civita symbol. The permutation symbol is a function of permutations of the numbers 1, 2, and 3. If the permutation is even, the function is positive. If the permutation is odd, the function is negative. If the permutation is neither even nor odd the function is zero. In three dimensions, the permutation symbol is given formally by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2) \text{ or } (2,1,3) \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

One of the most common uses of the permutation symbol is to simplify the expression for the cross product

$$\mathbf{A}\times\mathbf{B} = \det\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k$$

Using the Einstein summation convention, the expression simplifies to

$$\mathbf{A}\times\mathbf{B} = \varepsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k$$

It is also possible to write the curl of a vector field in terms of the permutation symbol as

$$\nabla\times\mathbf{F} = \varepsilon_{ijk}\hat{\mathbf{e}}_i\frac{\partial}{\partial x^j}f_k$$

where f_k is the k-th component of the vector field $\mathbf{F}$

2.5 The metric tensor

A special kind of tensor known as the *metric tensor* , commonly denoted as g_{ij} appears during formulations of problems in multiple coordinate systems. The metric tensor gives a way to measure distances in any given coordinate system. In three dimensions, the metric tensor is a 3x3 matrix which is a diagonal matrix for

most commonly used coordinate systems. The general form of the metric tensor is

$$g_{ij} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

The diagonal components of the metric tensor are referred to as *scale factors*. The term g_{11} refers to the scale factor for the first coordinate, g_{22} refers to the scale factor for the second coordinate, and so on. For example, in the Cartesian coordinate system which is described by x, y and z , $g_{11} = g_{xx}, g_{22} = g_{yy}$ and $g_{33} = g_{zz}$. The scale factors for the Cartesian coordinate system are all equal to one and so the metric tensor in the Cartesian coordinate system is equivalent to the Kronecker delta or the 3x3 identity matrix. With the introduction of curvilinear coordinate systems such as cylindrical or spherical, the scale factors change. For example, in orthogonal cylindrical coordinates, the metric tensor becomes

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However in using an orthonormal basis for cylindrical coordinates, the metric tensor can be represented by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse metric tensor, denoted by g^{ij} is given by

$$g^{ij} = \begin{pmatrix} \frac{1}{g_{11}} & 0 & 0 \\ 0 & \frac{1}{g_{22}} & 0 \\ 0 & 0 & \frac{1}{g_{33}} \end{pmatrix}$$

The metric tensor can be used to derive formulas in a variety of different coordinate systems in a systematic fashion using concepts explained in the next section.

2.6 Tensor derivatives and the covariant derivative

In order to arrive at the governing equations of elasticity, tensor derivatives must be introduced along with the associated notation. In tensor calculus, basic derivatives are denoted by a comma i.e

$$f_{,x} = \frac{\partial f}{\partial x}, \quad f_{,xy} = \frac{\partial^2 f}{\partial y \partial x}$$

Thus for a first or second rank tensor, the comma derivative is

$$T_{i,j} = \frac{\partial T_i}{\partial x^j}, \quad T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k}$$

In the derivation of the governing equations of elasticity for a homogeneous material, it is more convenient to use basic derivatives acting on the displacement components of the strain tensor. When deriving the equations for a nonhomogeneous material (such as a functionally graded material), it is more convenient to employ what is known as a *covariant derivative* which is a special kind of derivative use for tensors on the stress tensor. The covariant derivative has more terms that must be included than the basic derivative. The extra terms in the covariant derivative are known as *Christoffel symbols of the second kind.* Christoffel symbols, denoted by Γ with two subscripts and one superscript describe how the local coordinate bases change from point to point. They are defined by

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{mk}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m})$$

Where g is the metric tensor of the given coordinate system. The covariant derivative is usually denoted by a semicolon or a nabla (∇). For a rank one tensor, the covariant derivative is either defined as

$$T^i_{;j} = T^i_{,j} + \Gamma^i_{jk} T^k$$

$$T_{i;j} = T^i_{,j} - \Gamma^k_{ij} T_k$$

For a second rank covariant tensor, the covariant derivative is given by

$$T_{ij;k} = T_{ij,k} - \Gamma^m_{ji} T_{mj} - \Gamma^m_{kj} T_{im}$$

As the Christoffel symbols are calculated using the derivatives of the scale factors in the metric tensor, for Cartesian coordinates, each Christoffel symbol is zero as each scale factor is equal to one. In curvilinear

coordinates, nonzero terms arise when differentiating the scale factors. For example, in cylindrical coordinates, there are three nonzero Christoffel symbols. $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$. The nonzero Christoffel symbols appear when deriving equations in different coordinate systems. For example, Newton's law in Cartesian coordinates is

$$\sum \mathbf{F} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y + \ddot{z}\hat{e}_z$$

While in cylindrical coordinates it becomes

$$\sum \mathbf{F} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{e}_z$$

2.7 Derivation of the governing equations of elasticity

With the appropriate methods of tensor calculus developed, we are now able to derive the the governing equations of elasticity. Applying a force balance to an elastic medium,

$$\sigma_{ij,j} + F_i = \rho u_{i,tt} \tag{2.6}$$

where σ_{ij} is the three dimensional Cauchy stress tensor, F_i is the body force vector acting on the medium, and u_i is the i-th component of the displacement vector. In the static case such as the one being considered in this paper, the right hand side of equation (5) is equal to zero and leaves

$$\sigma_{ij,j} = -F_i \tag{2.7}$$

In order to unpack equation 2.7, the components of the stress tensor must be expanded. The components of the stress tensor as given in Green & Zerna [11] are

$$\sigma_{ij} = C^{kl}_{ij}\varepsilon_{kl}$$

Where C^{kl}_{ij} is the fourth order elasticity tensor and ε_{kl} is the second order strain tensor. The components of the elasticity tensor are given by

$$C^{kl}_{ij} = \lambda\delta_{ij}\delta^{kl} + \mu\left(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j\right)$$

Where λ and μ are known as *Lame's parameters*. For a homogeneous media, Lame's parameters are taken to be constant while in FGM's, the Lame's parameters vary with position. The components of the strain

tensor are

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Expanding out all terms

$$\sigma_{ij} = \lambda\delta_{ij}\delta^{kl}\varepsilon_{kl} + \mu\left(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k\right)\varepsilon_{kl}$$

Using the properties of the Kronecker delta function detailed in section 2.3, the components of the stress tensor can finally be written as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} = \lambda\delta_{ij}u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right) \tag{2.8}$$

differentiating with respect to j yields

$$\sigma_{ij,j} = \lambda\delta_{ij}u_{k,kj} + \mu\left(u_{i,jj} + u_{j,ji}\right)$$

Using the index changing properties of the Kronecker delta, the term $\delta_{ij}u_{k,kj}$ can be changed to $u_{k,ki}$. Next, we replace the dummy indices k, k with j, j

$$\sigma_{ij,j} = \lambda u_{j,ji} + \mu\left(u_{i,jj} + u_{j,ij}\right)$$

Collecting like terms and swapping the i and j indices on the last term, we arrive at the final vector form of the governing equation of elastostatics.

$$\mu u_{i,jj} + \left(\lambda + \mu\right)u_{j,ji} + F_i = 0 \tag{2.9}$$

Equation 2.9 which is often referred to as the Cauchy-Navier equation can be expressed in terms of vector operators as

$$\mu\nabla^2\mathbf{u} + \left(\lambda + \mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + F_i = 0 \tag{2.10}$$

where ∇^2 is the vector Laplacian, $\mathbf{u}$ is the displacement vector which has three components, and $\nabla\left(\nabla\cdot\right)$ is the gradient of the divergence of the displacement vector. This equation decomposes into three separate equations which when solved, yield the three components of the displacement vector. In cylindrical coordinates, equation 2.10 becomes the following

$$\left(\lambda + 2\mu\right)\left(u_{,rr} + \frac{1}{r}u_{,r} - \frac{1}{r^2}u\right) + \mu u_{,zz} + \left(\lambda + \mu\right)w_{,rz} = -F_r \tag{2.11}$$

$$v_{,rr} + \frac{1}{r}v_{,r} - \frac{1}{r^2}v + v_{,zz} = -F_\theta \tag{2.12}$$

$$(\lambda + \mu)\left(u_{,rz} + \frac{1}{r}u_{,z}\right) + \mu\left(w_{,rr} + \frac{1}{r}w_{,r}\right) + (\lambda + 2\mu)\, w_{,zz} = -F_z \tag{2.13}$$

In equations 2.11-2.13, u is the displacement in the r direction, v is the displacement in the θ direction, and w is the displacement in the z direction. If the forces are taken to be unit impulse forces represented by Dirac delta functions, then the solutions of equations 2.11-2.13 become the Green's functions for a homogeneous cylindrical solid.

2.8 Helmholtz decomposition

While generally the three components of equation 2.10 are solved instead of the entire vector equation, there is a method that simplifies the vector equation and allows for easier analysis. The method, known as *Helmholtz decomposition* and is discussed in more detail in [13] simplifies the vector equation by assuming a solution composed of the gradient of a scalar field and the curl of a vector field. More formally, for equation 2.10, a Helmholtz decomposition assumes the form

$$\mathbf{u} = \nabla\varphi + \nabla \times \psi \tag{2.14}$$

Where φ is a scalar function and ψ is a vector field. Inserting 2.14 into 2.10 gives

$$\mu\nabla^2\left(\nabla\varphi + \nabla \times \psi\right) + (\lambda + \mu)\,\nabla\left(\nabla \cdot \left(\nabla\varphi + \nabla \times \psi\right)\right) = -F_i \tag{2.15}$$

Two vector identities that can be used to simplify the equation are

$$\nabla \cdot \nabla\varphi = \nabla^2\varphi$$

and

$$\nabla \cdot \nabla \times \psi = 0$$

With theses identities, equation 2.15 becomes

$$\nabla\left[(\lambda + 2\mu)\,\nabla^2\varphi + F\right] + \nabla \times \left[\mu\nabla^2\psi + \mathbf{F}\right] = 0 \tag{2.16}$$

in order for equation 2.16 be satisfied, it is necessary that

$$\nabla^2 \varphi = -\frac{1}{c_s^2} F \tag{2.17}$$

$$\nabla^2 \psi = -\frac{1}{c_v^2} \mathbf{F} \tag{2.18}$$

where $c_s^2 = \lambda + 2\mu$ and $c_v^2 = \mu$. Equations 2.17 and 2.18 constitute uncoupled Poission equations. While the solutions of Poisson's equation are well understood, equations 2.17 and 2.18 are coupled through boundary conditions which still produces significant mathematical complications.

Chapter 3

Governing equations for an inhomogeneous solid

The equations derived in section 2.7 were for those for a homogeneous material. With the introduction of functionally graded material, the governing equations of elasticity must be re-derived for nonhomogeneous materials. While the equations in section 2 were derived using the displacement formulation, it is more convenient for the nonhomogeneous case to start from the stress formulation of the governing equation and applying a covariant derivative to the stress tensor. The result of applying a force balance and using stress components instead of displacement components is given by Sokolnikoff [12] and is of the form

$$\tau_{rr,r} + \tau_{rz,z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = -F_r \tag{3.1}$$

$$\tau_{r\theta,r} + \tau_{\theta z,z} + \frac{2}{r}\tau_{r\theta} = -F_\theta \tag{3.2}$$

$$\tau_{rz,r} + \tau_{zz,z} + \frac{1}{r}\tau_{rz} = -F_z \tag{3.3}$$

In equations 3.1-3.3, $\tau_{rr},\ \tau_{rz},\ \tau_{\theta z},\ \tau_{zz},\ \tau_{r\theta}$ and $\tau_{\theta\theta}$ are components of the stress tensor. Each component is defined in terms of the displacements $u,\ v,$ and w as

$$\tau_{r\theta} = \mu\left(v_{,r} - \frac{v}{r}\right), \qquad \tau_{\theta z} = \mu v_{,z}$$

$$\tau_{rr} = (\lambda + 2\mu)\, u_{,r} + \frac{\lambda}{r} u + \lambda w_{,z}$$

$$\tau_{\theta\theta} = \lambda u_{,r} + \frac{\lambda + 2\mu}{r} u + \lambda w_{,z}$$

$$\tau_{zz} = \lambda u_{,r} + \frac{\lambda}{r} u_{,r} + (\lambda + 2\mu)\, w_{,z}$$

$$\tau_{rz} = \mu\,(u_{,z} + w_{,r})$$

Here, instead of λ and μ being constants, they are taken to vary exponentially in the r and z direction as

$$\mu = \mu_o \exp\left(k_r(r/r_o) + k_z(z/z_o)\right) \tag{3.4}$$

$$\lambda = \lambda_o \exp\left(k_r(r/r_o) + k_z(z/z_o)\right) \tag{3.5}$$

Where μ_o and λ_o are reference Lame's parameters and k_r and k_z represent grading parameters in the r and z direction respectively. The terms r_o and z_o represent reference lengths in the r and z directions respectively and are not equal to zero. After performing the necessary derivatives of the stress components and substituting into equations 3.1-3.3, the resulting equations become unwieldy. In order to simplify the equations, $k_r,\ k_z,\ k_\theta,\ B_R,\ B_z,$ or B_θ can be set to zero. Application of this leads to four sets of coupled equations for u and w and two equations for v. For example, when k_r is set to zero, the governing equation for the torsional displacement becomes.

$$v_{,rr} + \frac{v_{,r}}{r} - \frac{v}{r^2} + \frac{k_z}{z_o} v_{,z} + v_{,zz} = -\frac{B_\theta}{2\pi r \mu_o} \exp\left(-k_z(z/z_o)\right) \delta(r-a)\delta(z) \tag{3.6}$$

and in the case where $k_z = 0$

$$v_{,rr} + \frac{v_{,r}}{r} - \frac{v}{r^2} + \frac{k_r}{r_o} v_{,r} - \frac{k_r}{r_o}\frac{v}{r} + v_{,zz} = -\frac{B_\theta}{2\pi r \mu_o} \exp\left(-k_r(r/r_o)\right) \delta(r-a)\delta(z) \tag{3.7}$$

and for the u and w equations with $k_r = 0$ and $B_r = 0$

$$\gamma^2 \left(u_{,rr} + \frac{1}{r} - \frac{1}{r^2} u\right) + u_{,zz} + \frac{k_z}{z_o} u_{,z} + (\gamma^2 - 1) w_{,rz} + \frac{k_z}{z_o} w_{,r} = 0 \tag{3.8}$$

$$(\gamma^2 - 1)\left(u_{,rz} + \frac{1}{r} u_{,z}\right) + \frac{k_z}{z_o}(\gamma^2 - 1)\left(u_{,r} + \frac{1}{r} u\right) + w_{,rr} + \frac{1}{r} w_{,r} + \gamma^2 \left(w_{,zz} + \frac{k_z}{z_o} w_{,z}\right) = -\frac{B_z}{2\pi r \mu_o} \exp\left(-k_z(z/z_o)\right) \delta(r-a)\delta(z) \tag{3.9}$$

where, $\gamma^2 = \mu_0/(\lambda_0 + 2\mu_0)$ The three other sets of equations for u and w with different parameters set to zero

are included in the appendix. If superposition is used, solving each set of equations would yield the Green's function for the exponentially graded cylinder. λ and μ need not vary exponentially however, functionally graded materials can vary linearly or quadratically as well and have been considered in [16]. In the two following sections, the limitations of standard techniques for solving partial differential equations will be shown.

Chapter 4

Separation of Variables and Special Functions

The purpose of this section is to provide a brief explanation of the solutions of equations 2.11 and 2.13 which were developed by Filon [9]. The main technique used in the solution method is known as *separation of variables*. In order to invoke separation of variables, first, equation 2.11 is differentiated with respect to z and equation 2.13 is differentiated with respect to r. Performing this yields

$$(\lambda + 2\mu)\left(u_{,rrz} + \frac{1}{r}u_{,rz} - \frac{1}{r^2}u_{,z}\right) + \mu u_{,zzz} + (\lambda + \mu)\, w_{,zzr} = 0 \tag{4.1}$$

$$(\lambda + \mu)\left(u_{,zrr} + \frac{1}{r}u_{,zr}\right) + \mu\left(w_{,rrr} + \frac{1}{r}w_{,rr}\right) + (\lambda + 2\mu)\, w_{,zzr} = 0 \tag{4.2}$$

Equations 4.1 and 4.2 are then written in terms of two different differential operators, $\mathscr{D}^2$ and D, where $\mathscr{D}^2 := \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r$ and $D := \frac{d}{dz}$. With these operators defined, equations 4.1 and 4.2 become

$$\left((\lambda + 2\mu)\,\mathscr{D}^2 + \mu D^2\right)\frac{\partial u}{\partial z} + (\lambda + \mu)\, D^2 \frac{\partial w}{\partial r} = 0 \tag{4.3}$$

$$(\lambda + \mu)\,\mathscr{D}^2 \frac{\partial u}{\partial z} + \left(\mu\mathscr{D}^2 + (\lambda + 2\mu)\, D^2\right)\frac{\partial w}{dr} = 0 \tag{4.4}$$

It can be shown that $\partial u/\partial z$ and $\partial w/\partial r$ satisfy the partial differential equation

$$\left(\mathscr{D}^2 + D^2\right)^2 y = 0 \tag{4.5}$$

In order to solve equation 4.5, separation of variables can be used. Separation of variables assumes a solution of the form

$$y(r,z) = R(r)Z(z) \tag{4.6}$$

Inserting 4.6 into the original partial differential equation yields two uncoupled ordinary differential equations. Upon substituting 4.6 into 4.5, one obtains the ordinary differential equations

$$\left(\mathscr{D}^2 - k^2\right) R = 0 \tag{4.7}$$

and

$$\left(D^2 - k^2\right) Z = 0 \tag{4.8}$$

Where k^2 is known as the separation constant. The equation 4.7 has the solutions

$$I_1(kr),\ \ K_1(kr),\ \ rI_0(kr),\ \ rK_0(kr)$$

Where I_ν and K_ν are modified Bessel functions of the first and second kind I.E they are solutions to the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\nu^2}{r^2} - 1\right) v(r) = 0 \tag{4.9}$$

The explicit forms of I_ν and K_ν are

$$I_\nu(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{2k+\nu}}{k!\Gamma(\nu + k + 1)}, \quad K_\nu(r) = \frac{\pi}{2}\frac{I_{-\nu}(r) - I_\nu(r)}{\sin(\pi\nu)}$$

The solutions of 4.8 are

$$z\cos(kz) \ \text{ and } z\sin(kz)$$

With both $R(r)$ and $Z(z)$ found, we can insert the solutions into 4.6 for the final form of y.

$$y = A\cos(kz + \alpha)I_1(kr)$$

$$B\cos(kz + \beta)K_1(kr)$$

$$C\cos(kz + \gamma)rI_0(kr)$$

$$D\cos(kz + \delta)rK_0(kr)$$

$$Ez\cos(kz+\varepsilon)I_1(kr)$$

$$Fz\cos(kz+\theta)K_1(kr)$$

In order to obtain the solutions to u and w, each form of y can be integrated to find the general form of the solution and then the constants can be determined from boundary and loading conditions.

Chapter 5

Integral Transforms

While separation of variables is a viable method for equations 2.11 and 2.13 in cylindrical coordinates for a homogeneous material, another method of solving differential equations known as integral transforms is used for either different coordinate systems or for inhomogeneous materials. This section will provide some background information on integral transforms, how integral transforms have been used to solve problems in linear elasticity, and the limitations of solving equations of integral transforms.

5.1 General theory and usage

The purpose of integral transforms is to transform an ordinary or partial differential equation into either an algebraic equation or a differential equation that is easier to solve. Once the transformed equation is solved, the solution is inverted back to the original domain to yield a solution to the original differential equation. There are different kinds of integral transforms such as the Fourier transform, the Laplace transform, the Hankel transform, and others. Each transform has the same fundamental form. Given a function $f(t)$, the transform of that function is given by

$$(Tf)(s) = \int_{t_1}^{t_2} K(t,s) f(t)\ dt \tag{5.1}$$

The integral in equation 5.1 produces a function Tf which is the transformed function. If an inverse integral exists, then the inverse of Tf is

$$f(t) = \int_{s_1}^{s_2} K^{-1}(t,s)(Tf)(s)\ ds \tag{5.2}$$

in equations 5.1 and 5.2, $K(t,s)$ is known as the *kernel* of the integral transform and is different for each specific integral transform. For example, in the Fourier transform $K(s,t) = e^{-2\pi ist}$. The bounds t_1 and t_2 also vary for most integral transforms; however, the bounds are usually 0 and ∞ or $-\infty$ and ∞. For the Fourier transform, the bounds are $-\infty$ and ∞. Combining the bounds and the kernel, the formal definition of the Fourier transform becomes

$$\mathcal{F}\{f(t)\} = \bar{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\, dt$$

Another widely used integral transform is known as the Hankel transform. Unlike other transforms, the Hankel transform can take on different orders. A ν-th order Hankel transform is defined as

$$\mathscr{H}_\nu\{f(t)\} = \hat{f}(s) = \int_0^{\infty} f(t)J_\nu(st)s\, dt$$

where, J_ν is a ν-th order Bessel function which is a solution to the ordinary differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

One of the most important properties of integral transforms is that most integral transforms are defined for the derivatives of functions. For most transforms, the transform of the derivative of a function is an algebraic term instead of a derivative which causes the derivative term to vanish. For example, for the Laplace transform, the transform of the derivative of a function is

$$\mathscr{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

This property allows for differential equations to be transformed into simple algebraic equations which can be easily solved. For example, consider the differential equation

$$\frac{dy}{dt} + y(t) = 0 \tag{5.3}$$

with the initial condition $y(0) = 1$. Applying the Laplace transform yields

$$sY(s) - 1 + Y(s) = 0$$

Solving for $Y(s)$

$$Y(s) = \frac{1}{s-1}$$

The inverse Laplace transform of $Y(s)$ is the solution of the differential equation $y(t)$. In order to invert the transformed solution, the direct inversion integral can be used or a table of Laplace transforms can be used to find the appropriate inversion. In this case the solution $y(t) = e^{-t}$.

5.2 Solution of governing equations for inhomogeneous media

Integral transforms have proved extremely useful in the solving of both ordinary and partial differential equations. As an example, the torsional Green's function for an exponentially graded cylinder was found by Mokashi [8]. Two different equation were solved for the case when the graded was in the longitudinal direction and when the grading was in the radial direction. In the case where there is grading in the longitudinal direction, the equation is

$$v_{,rr} + \frac{v_{,r}}{r} - \frac{v}{r^2} + \frac{k_z}{z_o} v_{,z} + v_{,zz} = -\frac{B_\theta}{2\pi r \mu_o} \exp\left(-k_z(z/z_o)\right) \delta(r-a)\delta(z) \tag{5.4}$$

This equation can be solved using the Hankel transform which was defined above. The first three terms on the left hand side of equation 5.4 are known as the Bessel operator and has the first order Hankel transform [17]

$$\mathscr{H}_1 \left\{ v_{,rr} + \frac{1}{r} v_{,r} - \frac{1}{r^2} v \right\} = -s^2 \hat{v}$$

Applying a first order Hankel transform with respect to r to the rest of equation 5.2.1 yields

$$\left(-s^2 + \frac{k_z}{z_o} \frac{d}{dz} + \frac{d^2}{dz^2} \right) \hat{v} = \frac{-B_\theta}{2\pi\mu_o} J_1(sa) \exp\left(-k_z(z/z_o)\right) \delta(z) \tag{5.5}$$

The transformed equation 5.5 is a simple linear second order differential equation with constant coefficients. In order to solve for the constants, the two conditions that must be satisfied are

$$\hat{v}^+ = \hat{v}-$$

$$\frac{d\hat{v}^+}{dz} - \frac{d\hat{v}^-}{dz} = -\frac{B_\theta}{2\pi\mu_o} J_1(sa)$$

After solving the transformed equation and solving for the constants,

$$\hat{v}(s,z) = \frac{-B_\theta}{2\pi\mu_o} J_1(sa) \frac{1}{\sqrt{s_o^2+s^2}} \exp\left(-s_o z\right) \exp\left(-\sqrt{s_o^2+s^2}|z|\right)$$

where $s_o = \frac{k_z}{2z_o}$.In order to obtain the solution to the original equation, the transformed solution must be inverted using the inversion integral. Upon applying the inversion integral the solution for the equation can be obtained

$$u(r,z) = \frac{B_z}{4\pi\mu_o} \exp\left(-s_o z\right) \int_0^\infty \frac{s}{\sqrt{s_o^2+s^2}} J_1(sa) J_1(rs) \exp\left(-\sqrt{s_o^2+s^2}|z|\right)\ ds \tag{5.6}$$

Further properties of Bessel functions can be used to simplify integral 5.6 into the finite integral

$$u(r,z) = \frac{B_z}{4\pi^2\mu_o} \int_0^\pi \frac{\cos\phi}{\sqrt{a^2+r^2+z^2-2ar\cos\phi}} \exp\left[-s_o\left(\sqrt{a^2+r^2+z^2-2ar\cos\phi}+z\right)\right] d\phi \tag{5.7}$$

This finite integral makes for easier numerical calculations. The displacement solutions are finite at $r = 0$ and there is a singularity at $r = a$

5.3 Limitations

Despite the utility of integral transforms shown in the preceding subsections, there are inherent limitations which prevent the solutions certain differential equations like 2.11 and 2.13 or in the inhomogeneous case 3.8 and 3.9. One of the biggest limitations of integral transforms is the inability to solve non-linear differential equations. While the integral transform for a function $f(t)$ may be defined, the integral transform for $(f(t))^2$ or $\sin(f(t))$ is usually not defined and prevents the solution of non-linear differential equations. Beyond non-linearities, integral transforms for derivatives or division by the dependent variable may lead to difficulties when applying integral transforms. For example, applying a Hankel transform of the derivative of a function, one obtains

$$\mathscr{H}_\nu\left\{\frac{df}{dr}\right\} = s\left(\frac{\nu+1}{2\nu}\hat{f}_{\nu-1}(s) - \frac{\nu-1}{2\nu}\hat{f}_{\nu+1}\right) \tag{5.8}$$

This implies that the order of the transform will change and can change more than once. This presents problems when attempting to solve equations 3.8 and 3.9 by integral transforms. Specifically, applying a first order Hankel transform to the $w_{,rz}$ term in equation 3.8 would yield an $s\hat{w}_{0,z}$ term. However, applying the same first order Hankel transform in to the $w_{,zz}$ term in equation 3.9 would yield $\hat{w}_{1,zz}$. This implies that there would be at least two different $\hat{w}$ terms along with at least one $\hat{u}$ term. As there are two transformed

equations and at the very least three unknowns, the transformed equation is unable to be solved via the combination of the Hankel and Fourier transform.

This result brings into question some published solutions in elastodynamics. Rahman [14] proposed solutions to the equations 2.11 and 2.13 with time dependent body forces acting on the medium. The solution method detailed was that of a mixed Fourier-Hankel transform. Specifically, Rahman uses a Fourier transform on both equations and also a zero order Hankel transform on each w term and a first order Hankel transform on each u term. However as seen by equation 5.8, applying a zero order transform to the $w_{,rz}$ term in equation 2.11 would cause complications as the zero-th order Hankel transform for the derivative of a function is undefined. Beyond this, in equation 2.13, Rahman claims to apply a first order Hankel transform to the term u/r which has has the Hankel transform

$$\mathscr{H}_\nu \left\{ \frac{1}{r} f(r) \right\} = \frac{s}{2\nu} \left(\hat{f}_{v-1}(s) + \hat{f}_{v+1}(s) \right) \tag{5.9}$$

This would imply that taking the first order transform of a $\frac{1}{r}u$ term would yield a $\hat{u}_0$ and a $\hat{u}_2$ term. The inclusion of at least two different $\hat{u}$ terms as well as $\hat{w}$ terms results in at least three different variables and only two algebraic equations. Despite these complications, Rahman asserts the equations can be transformed and subsequently inverted.

5.4 First approximations using modified equations

As shown in the preceding section, the method of integral transforms is not sufficient for solving equations 3.8 and 3.9. However, in the case that some terms in the equations are assumed to not contribute, equations which can be solved using integral transforms can be found. In equation 3.9, the $w_{,rz}$ term was found to cause complications when performing integral transforms. If that term is assumed to not contribute and to be set to zero, the equation becomes

$$(\lambda + 2\mu) \left(u_{,rr} + \frac{1}{r} u_{,r} - \frac{1}{r^2} u \right) + \mu u_{,zz} = \frac{-B_r}{2\pi r \mu} \delta(r-a)\delta(z) \tag{5.10}$$

Equation 2.13 has many more terms which cause complications with Hankel transforms such as a $u_{,rz}$ term and a $\frac{1}{r}u$ term and a $\frac{1}{r}w_{,r}$ term. When canceling each of the non-agreeable terms in equation 2.13, the result is the second order ordinary differential equation

$$(\lambda + 2\mu)\, w_{,zz} = 0 \tag{5.11}$$

Equation 5.10 is able to be solved using integral transforms while equation 5.11 is a simple second order ODE. Taking the first order Hankel transform of equation 5.10

$$\mu \hat{u}_{,zz} - (\lambda + 2\mu)\, s^2 \hat{u} = \frac{-B_r}{2\pi\mu} J_1(sa)\delta(z) \tag{5.12}$$

Integrating both sides of equation 5.12 from $-\infty$ to ∞, something about ε, we can see equation 5.12 satisfies the jump conditions

$$\hat{u}^+ = \hat{u}^-$$

$$\frac{d\hat{u}^+}{dz} - \frac{d\hat{u}^-}{dz} = -\frac{B_r}{2\pi\mu_o} J_1(sa)$$

The homogeneous solution of equation 5.12 is

$$\hat{u} = c_1 e^{cs^2 z} + c_2 e^{-cs^2 z} \tag{5.13}$$

where, $c = \sqrt{(\lambda + 2\mu)/\mu}$. the transformed solution is

$$\hat{u} = -\frac{B_r J_1(sa)}{2\pi\mu cs} e^{-cs^2 z} \tag{5.14}$$

in order to obtain the solution in the original domain, the solution must be inverted. Direct application of the inversion integral yields

$$u = -\frac{B_r}{2\pi\mu c} \int_0^\infty \exp(-csz)\, J_1(sa) J_1(sr)\, ds \tag{5.15}$$

This integral is standard and is given in [15] as

$$\int_0^\infty x^{\lambda-1} e^{-\alpha x} J_\mu(\beta x) J_\nu(\gamma x)\, dx = \frac{\beta^\mu \gamma^\nu}{\Gamma(\nu+1)} 2^{-\nu-\mu} \alpha^{-\lambda-\mu-\nu} \sum_{m=0}^{\infty} \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\Gamma(\mu+m+1)} \times F(-m; -\mu-m; \nu+1; \frac{\gamma^2}{\beta^2}) \left(\frac{-\beta^2}{4\alpha^2}\right)^m \tag{5.16}$$

Where $F(-m; -\mu-m; \nu+1; \frac{-\beta^2}{4\alpha^2})$ is known a hypergeometric function which is a solution to the differential equation

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0 \tag{5.17}$$

In the inversion integral, $\mu = 1,\ \nu = 1\ \beta = a,\ \gamma = r,\ \lambda = 2,\ \alpha = cz$. Upon simplification, the final form of

u becomes

$$u(r,z) = \frac{B_r a}{8\pi\mu_o c}\left[r(cz)^{-3}\sum_{m=0}^{\infty}\frac{\Gamma(3+2m)}{m!\Gamma(2+m)} \times F(-m;-1-m;2;1)\left(\frac{-r^2}{4c^2z^2}\right)\right] \tag{5.18}$$

While canceling the term in equation 3.8 that caused complications with transforms still yields a somewhat meaningful solution, every terms except for one in equation 3.9 produces complications with integral transforms. In the case where each of those terms in canceled, the remaining equation is

$$w_{,zz} = 0 \tag{5.19}$$

this second order ODE has the solution $w = c_0 + c_1 z$. However, for the displacement to go to zero as z tends to infinity, c_1 must be zero which implies the displacement in the z direction is constant.

5.5 First Approximations for the Conhomogeneous Case

In the nonhomogeneous case, there are more terms which cause complications when attempting integral transforms. In equation 3.8 when there is grading in the longitudinal direction and a force in the z direction, there is an additional $w_{,r}$ term which cannot be transformed properly. If both the $w_{,rz}$ term and $w_{,r}$ term are assumed to not contribute, the equation becomes

$$\gamma^2\left(u_{,rr} + \frac{1}{r} - \frac{1}{r^2}u\right) + u_{,zz} + \frac{k_z}{z_o}u_{,z} = -\frac{B_z}{2\pi r\mu_o}\exp\left(-k_z(z/z_o)\right)\delta(r-a)\delta(z) \tag{5.20}$$

which is of the same form of equation 5.4 and as the boundary conditions are the same, the solution for this equation is given by the integral in equation 5.7

Chapter 6

Conclusion

The mathematical background and derivation of the governing equations of elastostatics has been presented. For the case of a homogeneous material, the equations were derived using the displacement components of the strain tensor. In the case of a functionally graded material, the governing equations were derived by applying the covariant derivative to the stress tensor and substituting the displacement components.

Solution methods for a homogeneous infinite solid in cylindrical coordinates under various types of shear and the torsional Green's function of a functionally graded cylinder under a body force were detailed. For a homogeneous material, separation of variables was used as the derivatives of the solutions satisfied a partial differential equation that can be solved using separation of variables. As the derivatives satisfied the partial differential equation that could be solved using separation of variables, the solution was then integrated to solve the original equation and then the constants were solved for. The torsional Green's function for a functionally graded cylinder was found using integral transforms. Grading in the longitudinal direction was solved using a Hankel transform and can be displayed in closed form as an integral. For grading in the radial direction, the integral solution is non-standard and difficult to evaluate and so perturbation methods were used in the case of small values for the grading parameter.

The limitations of conventional methods of solving partial differential equations have been shown when applied to the equation for the Green's function for an inhomogeneous elastic solid. Integral transforms such as the Hankel transform are unable to be used as the application results in more equations than unknowns. It is now uncertain whether the published solutions to similar partial differential equations by Mujibur Rahman are mathematically sound. Beyond Hankel transforms, Mellin transforms and Fourier sine and cosine

transforms were also attempted and yielded similar results.

For future work, Numerical solutions may be considered. Other methods of solving partial differential equations could yield a solution. Methods such as attempting to formulate the problem in terms of dual integral equations which have well understood solution techniques as in [18] might be of interest.

Appendix

Shown below are the rest of the governing equations for an exponentially graded cylinder with one body force set to zero and one grading parameter set to zero

$k_r = 0, \;\; B_z = 0$

$$\gamma^2 \left(u_{,rr} + \frac{1}{r} u_{,r} - \frac{1}{r^2} u \right) + u_{,zz} + \frac{k_z}{z_O} u_{,z} + (\gamma^2 - 1) w_{,rz} + \frac{k_z}{z_0} w_{,r} = -\frac{B_r}{2\pi r \mu_o} \exp\left(-k_z (z/z_o)\right) \delta(r-a)\delta(z) \quad (6.1)$$

$$(\gamma^2 - 1) \left(u_{,rz} + \frac{1}{r} u_{,z} \right) + \frac{k_z}{z_o} (\gamma^2 - 2) \left(u_{,r} + \frac{1}{r} u \right) + w_{,rr} + \frac{1}{r} w_{,r} + \gamma^2 \left(w_{,zz} + \frac{k_z}{z_o} w_{,z} \right) = 0 \quad (6.2)$$

$k_z = 0, \;\; B_r = 0$

$$\gamma^2 \left(u_{,rr} + \frac{1}{r} u_{,r} - \frac{1}{r^2} u \right) + \frac{k_r}{r_o} \left(\gamma^2 u_{,r} + \frac{\gamma^2 - 2}{r} u \right) + u_{,zz} + (\gamma^2 - 1) w_{,rz} + \frac{k_r}{r_o} (\gamma^2 - 2) w_{,z} = 0 \quad (6.3)$$

$$(\gamma^2 - 1) u_{,rz} + \left(\frac{k_r}{r_o} + \frac{\gamma^2 - 1}{r} \right) u_{,z} + w_{,rr} + \left(\frac{k_r}{r_o} + \frac{1}{r} \right) w_{,r} + \gamma^2 w_{,zz} = -\frac{B_z}{2\pi r \mu_o} \exp\left(-k_r (r/r_o)\right) \delta(r-a)\delta(z) \quad (6.4)$$

$k_z = 0, \;\; B_z = 0$

$$\gamma^2 \left(u_{,rr} + \frac{1}{r} u_{,r} - \frac{1}{r^2} u \right) + \frac{k_r}{r_o} \left(\gamma^2 u_{,r} + \frac{\gamma^2 - 2}{r} u \right) + u_{,zz} + (\gamma^2 - 1) w_{,rz} + \frac{k_r}{r_o} (\gamma^2 - 2) w_{,z} = -\frac{B_r}{2\pi r \mu_o} \exp\left((-k_r (r/r_o)\right) \delta(r-a)\delta(z) \quad (6.5)$$

$$(\gamma^2 - 1) u_{,rz} + \left(\frac{k_r}{r_o} + \frac{\gamma^2 - 1}{r} \right) u_{,z} + w_{,rr} + \left(\frac{k_r}{r_o} + \frac{1}{r} \right) w_{,r} + \gamma^2 w_{,zz} = 0 \quad (6.6)$$

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